ECON 7010 - Macroeconomics I

Fall 2015

Notes for Lecture #3

Today- Cake eating problem:

- $T > 2$
- $T = \infty$
- Stochastic cake eating

The problem:

- T Period Problem ($T < \infty$)
  - $V_T(w_1) = \max_{(c_1,...,c_T)} \sum_{t=1}^{T} \beta^t u(c_t), \forall w_1$
    - s.t. $w_1 = \sum_{t=1}^{T} c_t$
    - $u' > 0, u'' < 0, u'(0) = \infty$
    - Note that V_T is the value function for the T period problem
    - Note that discounting period one has no effect on the Euler and will help us with the recursive solution to the $T + 1$ period problem
    - This notation represents is called the sequence problem (Bellman equation another way to represent the dynamic programming problem)
  - Lagrangian: $L = \max_{(c_1,...,c_T)} \sum_{t=1}^{T} \beta^t u(c_t) + \lambda \left(w_1 - \sum_{t=1}^{T} c_t \right)$
  - FOC: $\beta^t u'(c_t) = \lambda, t = 1, ..., T$ (so there are T FOCs and T unknowns)
  - $\Rightarrow u'(c_t) = \beta u'(c_{t+1}), t = 1, ..., T - 1$ ($T - 1$ necessary conditions from the Eulers, $+1$ necessary condition that says $w_1 = \sum_{t=1}^{T} c_t$)
- Policy Function: (generated by a given β and $u(\cdot)$)
  - θ is parameter describing $u(\cdot)$
  - β = discount factor
  - $(\theta, \beta) \Rightarrow$ policy function and value function
    - $c_t(w_1), t = 1, ..., T \rightarrow$ optimal solution to the problem
- Value function
  - $V_T(w_1) = \sum_{t=1}^{T} \beta^t u(c_t(w_1)), \forall w_1$
    - put in the optimal c_t's and get the total value of the problem $\rightarrow$ the max utility
    - e.g., $u(c) = \frac{c^{1-\gamma}}{1-\gamma} \Rightarrow R(c) = \gamma$
  - Suppose you have solved the T-period problem and now you must solve the T+1 period problem, 2 options:
    1. Solve $\max_{c_1,...,c_T} \sum_{t=0}^{T} \beta^t u(c_t)$
       - s.t. $w_0 = \sum_{t=0}^{T} c_t$
    2. solve $\max_{c_0} u(c_0) + V_T(w_1) \rightarrow V_T$ is the value function for the T period problem
       - $w_1 = w_0 - c_0 \Rightarrow$ will look like the 2-period problem where the "2nd period" payoff is $V_T(w_1)$
       - $\Rightarrow$ this recursive formulation uses the principle of optimality

- $T+1$ Problem
  - $V_{T+1}(w_0) = \max_{\{c_0,...,c_T\}} \sum_{t=0}^{T} \beta^t u(c_t)$
    - s.t. $\sum_{t=0}^{T} c_t = w_0$
    - Note: If we didn't discount period 1 in the value function above, we would go from $t = 1$ to $T+1$ here, adding a term at the end rather than the beginning
  - We can also write the above as a Bellman Equation:
  - $V_{T+1}(w_0) = \max_{c_0} u(c_0) + V_T(w_0 - c_0)$
    - Note that there is no β in front of V_T because in our specification of V_T above we discounted the first period
  - $= \max_{w_1} u(\underbrace{w_0 - w_1}_{\substack{\text{Difference between} \\ \text{what I have now } (w_0) \\ \text{and what I leave} \\ \text{for tomorrow } (w_1)}}) + V_T(\underbrace{w_1}_{\substack{\text{we have already} \\ \text{found the solution to} \\ \text{this } \forall w_1}})$
    - This means that to solve the $T+1$ period problem, after solving the T period problem, we only have to find one thing - the optimal c_0 (i.e., $w_0 - w_1$)
    - Note: This is an application of the principal of optimality. Once we have a solution to V_T, we only need to maximize for the one additional period because we know that solution for V_T will give us the optimal choices for the next T periods.
  - FOCs for T+1 problem:
    - $\beta^t u'(c_t) = \lambda, t = 0, ..., T \to T+1$ FOCs
    - $u'(c_t) = \beta u'(c_{t+1}), t = 0, ..., T-1$ equations
    - $u'(c_0) = \underbrace{V_T'(w_0 - c_0)}_{\substack{\text{the value of the } T \\ \text{period optimization problem}}} = \lambda = \beta u'(c_1) = \beta^2 u'(c_2) = ...$
    - Recall the 2-period problem: (Actually, go through the envelope for the T period problem here)
      - $\frac{dV_2}{dw_1} = u'(c_1) = \beta u'(c_2) \to$ we found this from applying the envelope theorem
      - This means that the change in the value of the value function is equal to the direct effect of the change in w_1 on the marginal utility in the first period (because we are at an optimal choice for our policy function - so the indirect effect is zero)
    - Principle of Optimality:
      - When $T+1$ periods, I only need to change one thing - just choose the right thing for period 0, then you know that the next T periods will be optimized, because we did that problem

- <u>Example</u>: $u(c) = ln(c)$
  - $V_1(w_T) = ln(w_T), \forall w_T$ (draw time line from 0 to T and show how work backwards)
  - $V_2(w_{T-1}) = \max ln(w_{T-1} - w_T) + \beta ln(\underbrace{w_T}_{V_1(w_T)})$
  - FOC: $\frac{1}{w_{T-1} - w_T} = \frac{\beta}{w_T} \Rightarrow w_T = \left(\frac{\beta}{1+\beta}\right) w_{T-1}$
  - so you can find c_{T-1}: $c_{T-1} = w_{T-1} - w_T = \frac{w_{T-1}}{1+\beta}$
  - plugging this into the Bellman equation, we can solve for V_2:
  - $V_2(w_{T-1}) = ln\left(\frac{w_{T-1}}{1+\beta}\right) + \beta ln\left(\frac{\beta w_{T-1}}{1+\beta}\right)$
  - simplifying: $= \underbrace{ln\left(\frac{1}{1+\beta}\right) + \beta ln\left(\frac{\beta}{1+\beta}\right)}_{A_2} + \underbrace{(1+\beta)}_{B_2} ln(w_{T-1})$

- $= A_2 + B_2 ln(w_{T+1})$
- One can keep going, backwards, to find solutions:
- $V_3(w_{T-2}) = \max_{w_{T-1}} ln(w_{T-2} - w_{T-1}) + \beta V_2(w_{T-1})$
- $= \max_{w_{T-1}} ln(w_{T-2} - w_{T-1}) + \beta(A_2 + B_2 ln(w_{T-1}))$
- To Solve:
  * Find FOC's: $\frac{1}{w_{T-2}-w_{T-1}} = \beta \frac{B_2}{w_{T-1}} \Rightarrow w_{T-2} = \frac{w_{T-1}(1+\beta B_2)}{\beta B_2} \Rightarrow w_{T-1} = \frac{w_{T-2}\beta B_2}{(1+\beta B_2)}$
  * Find w_{T-1} as a function of w_{T-2}
  * Solve for $V_3(w_{T-2}) = A_3 + B_3 ln(w_{T-2})$
- With this approach to solve the problem, all that matters is what you start with $\rightarrow$ past consumption not important per se $\rightarrow$ only good in that it tells you what you have today
  * State of the system is defined by how much cake you start with

Infinite Horizon Problem; $T = \infty$

- The problem: $V(w) = \max_c u(c) + \beta V(w'), \forall w \in [0, \bar{w}]$
  - where $w' = w - c$ (primes will denote the next period - since we are dealing with an infinite horizon, we'll drop the time subscripts)
- Terminology:
  - state variable(s): represent the state of the system, it's what you need to know to make a decision $\rightarrow$ here we just need to know the about of cake (w)
  - control variables: what is chosen, here it is c, it is under the max argument
  - stationarity: no t's; there is no end, so there is no need to keep track of time, certain relationships don't change over time (e.g., preferences)
  - transition equation: equation describing the evolution of the state variable, tells the value of the state variable tomorrow as a function of the state variable today and the control variable; e.g., $w' = w - c$
  - policy function: specifies the control variables as a function of the state variables; it's time invariant (e.g., in $T = 2$, $w_1 - c_1 = w_2 \Rightarrow \underbrace{c_1}_{control} = \underbrace{w_1 - w_2}_{state}$)
  - value function: not indexed by time in infinite horizon problem: $V(w)$
  - functional equation: unknown $V(w)$, e.g., $V(w) = \max_c u(c) + \beta V(w'), \forall w \in [0, \bar{w}]$
    * Note: A functional equation relates a function to other function in an implicit way - by writing it as a function of itself at another point in time, say. These function can't be reduced to algebraic equations
    * Note: The functional equation for the value function is called a Bellman equation (it's Bellman's Principle of Optimality that is used to solve these problems recursively)
    * Note: Richard Bellman was an American mathematician in the 20th century who invented dynamic programming
- In infinite horizon problem, time is irrelevant
  - lose all subscripts, use primes (e.g., w') to indicate future variables
- In finite horizon, we don't care how much we have left in period $T + 1$
  - In fact, that's how we solved the finite T problem (used this condition to start our backwards induction)

- We do worry about the amount in "$T+1$" in the infinite horizon problem
    - Will solve problem by using a fixed point theorem

- The Problem:
  - $V(w) = \max_c u(c) + \beta V(w'), \forall w \in [0, \bar{w}]$
  - or $V(w) = \max_{0 \leq w' \leq w} u(w - w') + \beta V(w'), \forall w$
  - control: w' (control the future state variable since it's determined with choice of c)
  - state: w
  - could say that we got rid of the transition equation by substitution ($c = w - w'$)
    - envelope theorem makes it easier to work with the latter than the former (this means that we don't have to look at $\frac{\partial c}{\partial w}$)

- Terminology in ∞ horizon:
  - $V(w) = \max_{0 \leq w' \leq w} u(w - w') + \beta V(w'), \forall w$
    - Will need $0 \leq \beta < 1$ to have convergent series (and thus a solution)
    - The problem needs to be stationary so that time per se is not relevant (shown by no time index)
  - Unknown is the value function: $V(w), \forall w$
  - Policy function: $w' = p(w); c = \phi(w); \Rightarrow w' = w - \phi(w)$
    - These are the policy functions b/c they are what you choose $\rightarrow$ choosing the amount of cake to eat today determines the cake for tomorrow

- FOC in the ∞-horizon problem:
  - $u'(w - w') = \beta V'(w')$
  - where do we get $V'(w')$?
    - FOC says: $\frac{dV}{dw'} = -u'(w - w') + \beta V'(w') = 0$
    - $\frac{dV}{dw} = V'(w) = \underbrace{u'(w - w')}_{=\beta V'(w')} + \underbrace{[-u'(w - w') + \beta V'(w')]}_{=0,\text{by envelope condition}} \frac{dw'}{dw} = u'(w - w')$
    - what this means is that since $V(w')$ is the max of all future consumption $\forall w'$, the effect of getting more cake now on utility is only the direct effect on today's consumption (derivative of future years flat around max, so no change in marginal lifetime utility from that).
    - $V'(w') = u'(w' - w'')$... etc, $\rightarrow$ the marginal value of extra cake today (tomorrow) is just the marginal value of consumption today (tomorrow)
  - Thus the Euler can be written as: $u'(w - w') = \beta u'(w' - w'')$ - which looks just like what we've seen all along - that discounted, marginal utilities are equalized along the optimal consumption path
  - What can we say about the solution to this problem:
    - $u'(c) = \beta u'(c') < u'(c) \Rightarrow c > c'$
    - $u'(\phi(w)) = \beta u'(\phi(w - \phi(w))) \forall w$
    - since $c' = \phi(w'), w' = w - c = w - \phi(w)$

- Example: $u(c) = ln(c)$, Solve by conjecturing a value function
  - recall: $V_T(w) = A_T + B_T ln(w)$
  - now, for the infinite horizon problem, guess that $V(w) = A + Bln(w)$
    - Here we are using the "guess and verify method" to solve this problem
    - Note that we are making a good guess here

    * While it can be helpful, won't need intuition like this to solve on computer- any initial guess will do. We will discuss more later.
  - Test our guess: $A + Bln(w) = \max_{w'} ln(\underbrace{w - w'}_{\text{policy function}}) + \beta(A + Bln(w')), \forall w$
  - FOC: $\frac{1}{w-w'} = \frac{\beta B}{w'} \Rightarrow w' = \frac{\beta B}{1+\beta B}w \rightarrow$ this is the policy function
    * Which means $V(w) = ln\left(\frac{w}{1+\beta B}\right) + \beta(A + Bln\left(\frac{\beta B}{1+\beta B}w)\right), \forall w$
    * $= ln\left(\frac{1}{1+\beta B}\right) + ln(w) + \beta A + \beta Bln\left(\frac{\beta B}{1+\beta B})\right) + \beta Bln(w), \forall w$
    * $= \underbrace{ln\left(\frac{1}{1+\beta B}\right) + \beta A + \beta Bln\left(\frac{\beta B}{1+\beta B}\right)}_{A \rightarrow \text{Find } A} + (\underbrace{1+\beta B}_{B \rightarrow \text{find } B})ln(w), \forall w$
    * $B = 1 + \beta B \Rightarrow B = \frac{1}{1-\beta}$
    * The above then means that we can solve the policy function in terms of parameters: $w' = \frac{\beta B}{1+\beta B}w = \frac{1}{\frac{1}{\beta B}+1}w = \frac{1}{\beta\frac{1}{\frac{1}{1-\beta}}+1}w = \frac{1}{\frac{1-\beta}{\beta}+1}w = \frac{1}{\frac{1-\beta+\beta}{\beta}} = \frac{1}{\frac{1}{\beta}}w = \beta w = \beta w$ (same as T period problem)
    * Now that we have B in terms of β, we can plug in above and solve for A in terms of β... do this...
    * $A = ln\left(\frac{1}{1+\beta B}\right) + \beta A + \beta Bln\left(\frac{\beta B}{1+\beta B}\right)$
    * $A = ln\left(\frac{1}{1+\beta\frac{1}{1-\beta}}\right) + \beta A + \beta\frac{1}{1-\beta}ln(\beta)$
    * $A = ln\left(\frac{1}{\frac{1-\beta+\beta}{1-\beta}}\right) + \beta A + \beta\frac{1}{1-\beta}ln(\beta)$
    * $A = ln\left(\frac{1}{\frac{1}{1-\beta}}\right) + \beta A + \beta\frac{1}{1-\beta}ln(\beta)$
    * $A = ln(1-\beta) + \beta A + \beta\frac{1}{1-\beta}ln(\beta)$
    * $A - \beta A = ln(1-\beta) + \beta\frac{1}{1-\beta}ln(\beta)$
    * $A(1-\beta) = ln(1-\beta) + \beta\frac{1}{1-\beta}ln(\beta)$
    * $A = \frac{ln(1-\beta)+\beta\frac{1}{1-\beta}ln(\beta)}{1-\beta}$
  - Thus we have the solution to the value function: $V(w) = \frac{ln(1-\beta)+\beta\frac{1}{1-\beta}ln(\beta)}{1-\beta} + \frac{ln(w)}{1-\beta}$ – this guess works!
  - Should be able to prove the correct case (as above) works and that wrong cases of $V(w)$ don't
  - e.g., try $V(w) = Bln(w), V(w) = A + Bw \rightarrow$ do these work?

- Example: $u(c) = ln(c)$, Solve by conjecturing a policy function
  - Guess $w' = \underbrace{\gamma}_{constant} * w$, so $c = (1-\gamma)w$
  - in ln case $c = (1-\beta)w, w' = \beta w$
    * This comes from fact that $c = w - w' = w - \beta w = (1-\beta)w$
  - $c' = (1-\beta)w' = (1-\beta)(w-c) = (1-\beta)(w-(1-\beta)w) = (1-\beta)(1-\beta)w$
  - Show that this guess works (we know it does, but to show how to do with with guess at policy function):
    * FOC implies: $\frac{1}{c} = \frac{\beta}{c'}$
    * Thus: $\frac{1}{(1-\beta)w} = \frac{\beta}{(1-\beta)w'} = \frac{\beta}{(1-\beta)\beta w}$
    * Thus we find: $\frac{1}{(1-\beta)w} = \frac{\beta}{(1-\beta)\beta w} = \frac{1}{(1-\beta)w}$, which is true - the guess is correct

- Should be able to prove the correct case (as above) works and that wrong cases of $w' = \phi(w)$ don't
- e.g., try $w' = (1-\beta)w$, or $w' = \beta ln(w) \to$ do these work?
- Draw a graph with time on the x-axis and c on the vertical. Draw a downward sloping curve.
- You can think of the above sequence as coming from the policy functions ($c = \beta w$) for $ln(c) = u(c)$ or the stationary policy function $c = \phi(w)$ or in terms of the Euler equation: $u'(c) = \beta u'(c') < u'(c) \Rightarrow c > c'$

- Finding a solution in general: $[V(w), \phi(w)]$
  - may have an example that you can work out analytically (as above)
  - more likely - need to compute the solution (we won't do today)
    - Will use a contraction mapping theorem and iteration
      - Proves that a sequence of functions will converge
      - $\{V_i(w)\} \to V(w)$
    - $V_{i+1}(w) = \max_{w'} u(w - w') + \beta V_i(w') \forall w$
    - $\{V_i(w)\} \to V(w)$
      - $V_i(w)$ is arbitrary
      - solution is a function $(V(w))$ and a policy function $(\phi(w))$
      - computer uses numbers on a grid to calculate (so it's discrete)

Adding uncertainty:

- Draw a graph with w on the x-axis and c on the y-axis. Draw some dots and a straight line of best fit through them. Call this the data.
  - Suppose $u(c) = ln(c)$, then we know $c = \beta w$
  - But data isn't just a straight line- it varies
  - can run a regression, $c = \beta w + \varepsilon$ and get an estimate of β, but we can't just add ε to our model
  - our functional equation for $V(w)$ before had no "wiggle room"
    - i.e., there is no error term in this equation, but there are in our data because there are unobservable variables/measurement error
- Since we have no error term in our previous models of $V(w)$, we need to add uncertainly to allow the empirical model to come together (with no uncertainly in the model, the data would reject it)
- Where to add uncertainty: (tildes indicate random variables)
  - tastes (i.e., in $u(\cdot)$): $\tilde{u}(c, \tilde{\varepsilon})$
  - storage technology: $w' = (w - c)\tilde{\rho}$
  - discount factor (e.g. probability of death/attrition): $\tilde{\beta}$
  - we'll work on just the first of these for now
- Choice under uncertainty
- Chapter 6 of Mas-Colell, Chapter 11 in Varian
- Now we'll start talking about preferences over lotteries
- Use expected utility theory
- Lottery

- wp = with probability
- Assume $c_H wp \pi$
- Assume $c_L wp(1-\pi)$
- Expected Utility
  * $EU = \pi u(c_H) + (1-\pi)u(c_L)$
  * this does not equal $u(\pi c_H + (1-\pi)c_L)$
  * for risk averse agents (i.e., those with a concave utility function ($u'' < 0$)), the utility of the expected value is higher than the expected utility of the lottery
  * i.e., the risk averse would rather take the average of the two with certainty than the prob of one or the other (b/c u concave, $u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$)
  * DRAW utility showing this

Cake eating with taste shock:

- Preferences: $u(c, \underbrace{\tilde{\varepsilon}}_{\text{taste shock}}) = \varepsilon u(c)$
  - normal assumptions on $u : u' > 0, u'' < 0$
  - Note that ε affects u and u'
- Storage Technology: $w' = w - c$
- Information: ε known when consumption choice is made (Important to specify timing of knowledge!!)
- Representing the shock: (distribution of the shock)
  - $\varepsilon = \{\varepsilon_L, \varepsilon_H\}$: high or low, ε has two values
  - First-Order Markov Process for the transition (this means that the probability of going to state X one period ahead only depends upon what state in today)
    * $Prob(\varepsilon_{t+1} = \varepsilon_j | \varepsilon_t = \varepsilon_i) = \pi_{ij}$ (Probability tomorrow = state j given today = state i)
    * Draw a 2x2 matrix with the transition probabilities between the high and low states
    * Called a first order process because what happens tomorrow only depends on where at today
  - IID process (independently and identically distributed)
    * Past doesn't matter (like drawing a random # each new day)
    * π_{ij} independent of i (constant $\rightarrow$ prob of state tomorrow does not depend on state today (Note: rows of matrix will still sum to one)